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Algorithmic Construction Of Equal Norm-Angle Tight Frames In R^2 Of A Given Vector Hilbert Space

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Abstract: The objective of this paper is to produce an implementable algorithm for construct equal norm equiangular tight frame (ENATF) in R^2 of a given vector. This structure has some application in some atomic structure in crystal physics and filter bank theory in communication.
Keywords : frame, equal norm, algorithmic, Hilbert space.

1. INTRODUCTION

Frames were introduced by Duffin and Shaefer in 1952 [6] as component in the development of nonharmonic Fourier series and a paper by Daubechies Grossman Meyer in 1986 initiated the use of frame theory in signal processing [3].The paper is organized as follows: section 2 contain preliminary definition on frames and the basic notation used throughout the paper. In section 3 we review some basic elements concerning the notion of equal norm-angle tight frame (ENATF) in the form suitable for the algorithm structure ENATF and we present throughout the review and several examples, and suggest some possible applications.

2. FINITE TIGHT FRAMES

Definition a sequence $\{w_i\}_{i=1}^{m}$ of elements of a Hilbert space n-dimensional H over C or R, is called finite frame, if there are constans A;B > 0 such that

$$Av \leq \sum_{i=1}^{m} < v, w_i > w_i \leq Bv.$$

For all $v \in H$. the numbers A;B are called frame bounds respectively. The frame is called tight frame if A = B. The tight frame is called Parse-val frame if A = 1. The frame is called equal norm frame if $||w_i|| = ||w_i||$ for all i; j . the frame is called equal angle if the angle betwen for $w_i;w_{i-1}$ or $w_i;w_{i+1}$ are equal for all i. The frame is called ENATF if it be a tight, equal norm and angular frame.
Some frames can be de_ned in a natural way by using group representation [2].
Theorem 2.1. [4] Let H be a real (complex) n - dimensional Hilbert space and G a finite group such that $g : H \to H$ be an unitary and irreducible representation and let $w \in H$ be a fixed vector. we defined the subgroup G_w of G as follows:
$G_w = \{g \in G | g(w) = \alpha(w)\}$
Where α is a scaler depending on g.if $\{g_i\}_{i=1}^{m}$ is a system of the left cosets of G on G_w then
$w_1 = g_1(w), w_2 = g_2(w), \dots , w_m = g_m (w)$
form an equal norm tight frame in H , namely for all $v \in H$ we have

$$\sum_{i=1}^{m} < v, w_i > \ w_i = \frac{m}{n} ||w||^2 \ v.$$

Corollary 2.2. The orbit G
$G(w) = \{g(w) | g \in G\}$
is a tight frame with bound $\frac{m'}{n}||v||^2$ where m' is a order the finite group G.
Proof. We refer to Theorem 2.1 we have for all $v \in H$

$$\sum_{i=1}^{m} < v, w_i > \ w_i = \frac{m}{n} ||w||^2 \ v.$$

The representation being unitary we have for any $g \in G$

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(2.1) $\langle v, w_i \rangle w_i = \langle v, g(w) \rangle g(w)$

Let k be the number of elements of Gw. Then G has m' = km elements, let {g^1; : : : ; $g^{m'}$} be an elements of the group G by equation 2.1 we have:

$$\frac{m}{v} ||v||^2 = m\langle v, g(w) \rangle g(w)$$

so $\frac{1}{n} ||w||^2 v = \langle v, g(w) \rangle g(w)$ consequently

$$\sum_{i=1}^{m'} \langle v, g^i(w) \rangle g^i(w) = \frac{m'}{n} ||w||^2 v$$

Proposition 2.3. let H = R^2 and $w^T = (\alpha 1; \alpha 2) \in$ H then a sequence $\{w_i\}_{i=0}^{m-1}$ (m >2) is a ENATF with angle $\frac{2\pi}{m}$ such that
w0 = $P^0 w; w^1 = P^1 w, \ldots, w_{m-1} = P^{m-1} w$
where $P^0 = I_{R^2}$,

$$P = \begin{bmatrix} \cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix}$$

namely for all v $\in$ H we have

$$\sum_{i=0}^{m-1} < v, w_i > w_i = \frac{m}{2} ||w||^2 v.$$

Proof. we consider the map g: $R^2 \rightarrow R^2$ where

$(\alpha_1, \alpha_2) \rightarrow (\alpha_1 \cos\frac{2\pi}{m} - \alpha_2 \sin\frac{2\pi}{m}, \alpha_1 \sin\frac{2\pi}{m} + (\alpha_2 \cos\frac{2\pi}{m}) = P_w$
the cyclic group $C_m = \langle g | g^m = e \rangle = \{e, g, \ldots, g^{m-1}\}$ defines a unitary and irreducible reperesentstion on R^2. so by 2.2 the vectors

w_0= w, w_1 = g(w) , … $w_{m-1} = g^{m-1}(w)$
are tight frame with bound $\frac{m}{2} ||w||^2$. under hand $g^i(w) = P^i$ w, since P is rotation of the R^2 so $\{w_i\}_{i=0}^{m-1}$ (m >2) is a ENATF frame with angle $\frac{2\pi}{m}$.

3. **ENATF algorithm**
Now we present the bellow algorithms for product ENATF frame with a favorite vector in R^2.
algoritm1. ENATF
parameter:
m number of desired frame vector
, $f_0 = x_0 e_0 + y_0 e_1$ is a given vector
algorithm:
2) For i= 1, …, M-1 do
4) $x_i = x_{i-1} \cos\frac{2\pi}{m} - y_{i-1} \sin\frac{2\pi}{m}$
5) $y_i = x_{-1i} \cos\frac{2\pi}{m} - y_{i-1} \sin\frac{2\pi}{m}$
6) $f_i = x_i e_1 + y_i e_2$
8) End.

output:
ENATF $\{f_i\}_{i=0}^{m-1}$.

Example 1. for m = 3 and f_0 = (1; 0) we have

$$w_0^T = (1,0), w_1^T = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), w_2^T = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

is a ENATF with bound $\frac{3}{2}$ and angle $\frac{2\pi}{3}$.

$$w_0^T = (1,0), w_1^T = (0,1), w_2^T = (-1,0), w_4 = (0,-1)$$

is a ENATF with bound 2 and angle $\frac{\pi}{2}$.
so for any m the algorithm product a m-regular hedral and such m $\rightarrow\infty$ this algoritm will prouduct a circle that is a continuous frame also for m = 3 the frames of algorithm product a Merseds Benz frame for use filter bank [1] and structer Honycomb lattic[5] . If m $\| v \|^2 = 2$ then the frames pruduct for algorithm is Parseval frame.

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